

Symplectic invariants, Virasoro constraints and Givental decomposition

*N. Orantin*¹

Département de mathématiques,
Chemin du cyclotron, 2
Université Catholique de Louvain,
1348 Louvain-la-Neuve, Belgium.

Abstract:

Following the works of Alexandrov, Mironov and Morozov, we show that the symplectic invariants of [14] built from a given spectral curve satisfy a set of Virasoro constraints associated to each pole of the differential form ydx and each zero of dx . We then show that they satisfy the same constraints as the partition function of the Matrix M-theory defined by Alexandrov, Mironov and Morozov. The duality between the different matrix models of this theory is made clear as a special case of dualities between symplectic invariants. Indeed, a symplectic invariant admits two decomposition: as a product of Kontsevich integrals on the one hand, and as a product of 1 hermitian matrix integral on the other hand. These two decompositions can be thought of as Givental formulae for the KP tau functions.

1 Introduction and main results

1.1 Introduction: Matrix M-theory

The theory of Hermitian random matrices is linked to many different fields in mathematics and physics by different very means (proved or conjectured) such as enumerative geometry, string theory or statistical physics. But matrix models are not only dual to different theories, they are also dual from one another, e.g. the hermitian one matrix model's partition function can lead to the Kontsevich integral by taking an appropriate limit of the moduli of this model [5]. More generally, it seems that there exist dualities between many different matrix models as it is pointed out in [2, 3, 4]. In this series of papers, the authors consider the possibility of the existence of a random matrix equivalent of M-theory, i.e. they claim, and give evidences, that there should exist a general "M-theory" whose partition function reduces to different kinds of matrix models in different patches of the moduli of this big theory². Even though such a general partition function was not explicitly built in these articles, they proposed to characterize it as the zero mode of a differential operator defined on an associated algebraic curve: the spectral curve. This global operator is shown to decompose as a sum of local Virasoro operators defined in the neighborhood of the singularities of the spectral curve.

Recent progresses in the resolution of hermitian random matrix models [10, 8, 12, 9, 14] have led to the definition of an infinite set of numbers $F^{(h)}$ associated to an arbitrary algebraic curve whether this algebraic curve comes from the study of a matrix model or not. More precisely, these works point

¹ E-mail: nicolas.orantin@cern.ch

²More precisely, this partition function is defined as the string theory partition function in the sense of [27].

out that the matrix models should not be studied only for themselves but are just *one representation of the fundamental symplectic integrable invariants in some very particular cases*. These more general partition functions always possess a topological expansion in a formal parameter N which is identified with the size of the matrices to be integrated in the matrix model representations. When the spectral curve is identified as coming from a matrix model³, this partition function is equal to the partition function of the matrix model considered. We are thus able to build a partition function associated to an arbitrary algebraic curve and which reduces to the partition function of matrix models for some particular curves: it is a good candidate for this "matrix M-theory" partition function.

Nevertheless, the link between the approach of Alexandrov, Mironov and Morozov [3] is not that obvious and it is an interesting problem to make it clear since both approaches can benefit from one another. This is precisely the aim of the present paper: we show that both definitions coincide since the recursion relations of [14] reduce to the Virasoro constraints of [3] when expressed in the right variables.

1.2 Main results

Let us consider an algebraic equation $\mathcal{E}(x, y)$ ⁴ represented by a compact Riemann surface Σ and two meromorphic functions x and y on it such that

$$\forall p \in \Sigma, \mathcal{E}(x(p), y(p)) = 0 \quad (1-1)$$

and the one-form ydx has poles α_i of degree $d_i + 1$ and dx has simple zeroes a_i . In the following, we refer to this equation as the *spectral curve*. Following [14], one defines the symplectic invariants $F^{(g)}(\mathcal{E})$ associated to this equation and build the partition function

$$\mathcal{Z}(\mathcal{E}) = e^{-\sum_g N^{2-2g} F^{(g)}(\mathcal{E})}. \quad (1-2)$$

The purpose of this paper is to find a set of differential operators annihilating this partition function.

One first shows, in theorem 3.2, that the "loop equations" satisfied by the correlation functions can be seen as Virasoro constraints annihilating the partition functions:

$$\widehat{\mathcal{L}}(p)\mathcal{Z} = 0 \quad (1-3)$$

with

$$\mathcal{L}(p) := \frac{1}{N^2} : \mathcal{J}^2(p) : + \sum_i \oint_{\alpha_i} \frac{: \mathcal{J}^2(q) :}{(z_i(q) - z_i(p)) dz_i(q)} \quad (1-4)$$

where the current is defined at any point p of the spectral curve by

$$\mathcal{J}(p) := Nydx(p) + \frac{1}{N} \partial_{B(.,p)} \quad (1-5)$$

with $\partial_{B(.,p)}$ the loop insertion operator of [14]. We then precise the link between the recursion relations of [14] and the Virasoro constraints of [3]. Indeed, we show that the recursive definition of the correlation functions is nothing but saying that the action of a Virasoro operator located at the branch points annihilates the partition function (see theorem 3.3):

$$\widehat{\mathcal{L}}(p)\mathcal{Z} = 0 \quad (1-6)$$

³There exists a generic procedure to associate a spectral curve to a given matrix model. For details see [14, 16] for example.

⁴The symplectic invariants can be built from more general plane curve but we restrict the study of this paper to algebraic curves. Indeed, one studies the variations of the symplectic invariants with respect to the holomorphic moduli of the spectral curve.

with

$$\widehat{\mathcal{L}}(p) := \sum_i \oint_{\alpha_i} \frac{dE_q^{(i)}(p)}{(y(q) - y(\overline{q}))dx(q)} : \mathcal{J}(q)\mathcal{J}(\overline{q}) : . \quad (1-7)$$

In section 4, we show that, in appropriate coordinates, these global Virasoro operators project to local operators

$$\mathcal{L}(p) \rightarrow \sum_{j=-1}^{+\infty} \frac{dz_i(p)}{z_i(p)^{j+1}} L_j^{(i)} \quad \text{as } p \rightarrow \alpha_i \quad (1-8)$$

with the *discrete Virasoro operators*

$$L_j^{(i)} = \frac{1}{N^2} \left(2j \frac{\partial}{\partial t_{j,i}} + \sum_{l=1}^{j-1} l(j-l) \frac{\partial^2}{\partial t_{j-l,i} \partial t_{l,i}} \right) + \sum_{k=1}^{d_i} (k+j) t_{j,i} \frac{\partial}{\partial t_{k+j,i}} \quad (1-9)$$

and a similar result for the projection of $\widehat{\mathcal{L}}(p)$ around the branch points.

Finally, these local Virasoro constraints allow us to decompose the partition function \mathcal{Z} as a product of one hermitian matrix integral on the one side and a product of Kontsevich integrals on the other side (see theorems 5.1 and 5.2):

$$\mathcal{Z}(\mathcal{E}) = e^{\mathcal{U}} \prod_i \mathcal{Z}_H(\mathbf{t}_i) = e^{\widehat{\mathcal{U}}} \prod_i \mathcal{Z}_K(\tau_i) \quad (1-10)$$

with two inter-twinning operators \mathcal{U} and $\widehat{\mathcal{U}}$ linking the global structure of the spectrale curve and the local behavior of ydx at its poles and its zeroes encoded in the moduli $t_{j,i}$ and τ_i respectively. These decomposition formulae were actually already derived by Givental in the study of the multi-component KP τ -function [19, 20] and observed by Chekhov in the matrix model framework in [7].

Remark 1.1 This last decomposition formula is proved only in the case of a genus 0 spectral curve. Indeed, if the spectral curve has higher genus, there exists an ambiguity in the definition of the terms of the decomposition which requires further investigations.

2 Symplectic invariants

Let us first summarize how the symplectic invariants and correlation functions are defined in [14] and introduce some useful notations.

2.1 Algebraic geometry: definitions and notations

In the following one considers an algebraic equation of degree $d_x + 1$ in x and $d_y + 1$ in y

$$\mathcal{E}(x, y) = 0 \quad (2-1)$$

referred to as *the spectral curve*. More precisely, the spectral curve is the triple (Σ, x, y) where Σ is a compact Riemann surface and $x(p)$ and $y(p)$ two meromorphic functions on it such that

$$\forall p \in \Sigma, \mathcal{E}(x(p), y(p)) = 0. \quad (2-2)$$

One also needs to equip this Riemann surface with a basis of canonical cycles $\{\mathcal{A}_i, \mathcal{B}_i\}_{i=1}^g$, where g is the genus of Σ .

2.1.1 Branch points and sheeted structure

Given a fixed value of x , the equation $\mathcal{E}(x, y)$ has generically $d_y + 1$ distinct solutions in y . This means that there exist $d_y + 1$ distinct points $p^i \in \Sigma$, $i = 0, \dots, d_y$, corresponding to the same value of x :

$$\forall (i, j) = 0, \dots, d_y, \quad x(p^i) = x(p^j). \quad (2-3)$$

This corresponds to saying that Σ can be viewed as $d_y + 1$ copies of the Riemann sphere, each corresponding to one particular p^i : one calls such a copy of \mathbb{CP}^1 , a y -sheet. But there also exist particular points a_i , $i = 1, \dots, \#bp$, where two pre-images of a given complex number $x(p)$ coincide:

$$\exists i \neq j, \quad p^i = p^j. \quad (2-4)$$

These points are called *x-branch points* since they correspond to loci where two y -sheets meet. They are solution of the equation

$$dx(a_i) = 0. \quad (2-5)$$

In the following, we always suppose that all the branch points are simple, i.e. they are simple zeroes of dx^5 . This restriction implies that as a point q approaches a branch point a_i , there exists a unique point \bar{q} such that $x(q) = x(\bar{q})$, $y(q) \neq y(\bar{q})$ and $\bar{q} \rightarrow a_i$ as $q \rightarrow a_i$. Remark that the application $q \rightarrow \bar{q}$ is not globally defined but only locally near each branch point⁶.

2.1.2 Fundamental differentials

We denote by $du_i(p)$ the g holomorphic differentials on Σ normalized on the \mathcal{A} -cycles:

$$\oint_{\mathcal{A}_i} du_j(p) = \delta_{ij}. \quad (2-6)$$

The *Bergman Kernel* $B(p, q)$ is the unique bidifferential having only one pole in p located at $p \rightarrow q$ such that

$$B(p, q) = \frac{dz(p)dz(q)}{(z(p) - z(q))^2} + \text{regular as } p \rightarrow q \quad (2-7)$$

and normalized by

$$\oint_{\mathcal{A}_i} B(p, q) = 0. \quad (2-8)$$

One also defines the *third Abelian differential*

$$dS_{p,p'}(q) = \int_p^{p'} B(q, \cdot) \quad (2-9)$$

which has two simple poles in $q \rightarrow p$ and $q \rightarrow p'$ with respective residues 1 and -1.

One especially needs a particular case of this differential when the integration path lies in the neighborhood of a branch point a_i linking one point p to its conjugate \bar{p} :

$$dE_p^{(i)}(q) := \frac{1}{2} \int_p^{\bar{p}} B(q, \cdot). \quad (2-10)$$

⁵ One can deal with higher order zeroes by merging such simple zeroes. It is studied in section 8 of [14].

⁶ Nevertheless it can be globally defined if the curve is hyperelliptic since it is the application which exchanges the y -sheets.

2.1.3 Moduli of the curve

The main differential involved in the theory of the symplectic invariants is the 1-form ydx . First of all, it has $\#_{poles}$ poles α_i of respective degrees d_i . It can thus be described by its behavior around these poles

$$\forall i = 1, \dots, \#_{poles}, \quad ydx(p) \sim \sum_{k=1}^{d_i} kt_{k,i} z_i^k(p) dz_i(p) \quad \text{as } p \rightarrow \alpha_i \quad (2-11)$$

where $z_i(p) = \frac{1}{\xi_i(z)}$ is the inverse of a local variable in the neighborhood α_i , i.e. it has a simple pole in α_i . It is build as follows: if x is regular at α_i , set $\xi_i(z) = x(z) - x(p)$, and if x has a pole of degree d at α_i , set $\xi_i(z) = x(z)^{-1/d}$. One also needs to precise its cycles integrals

$$\epsilon_i = \frac{1}{2i\pi} \oint_{\alpha_i} ydx. \quad (2-12)$$

The coefficients $t_{k,i}$ are called the *moduli at the pole* α_i and the cycle integrals ϵ_i are the *filling fractions*.

These moduli define totally the differential form ydx . Indeed, using the Riemann bilinear formula [17, 18], one can write it

$$ydx(p) = \sum_{i,k} kt_{k,i} B_{k,i}(p) + \sum_i t_{0,i} dS_{\alpha_i,o}(p) + 2i\pi \sum_i \epsilon_i du_i(p) \quad (2-13)$$

where

$$B_{k,i}(p) := - \operatorname{Res}_{q \rightarrow \alpha_i} B(p, q) z_i(q)^k. \quad (2-14)$$

2.2 Symplectic invariants, correlation function and free energy

We now have everything in hand to define the central objects of this theory.

2.2.1 Definitions

Following [14], let us define recursively the k points, genus h *correlation functions* $W_k^{(h)}(p_1, \dots, p_k)$ as a k -form by

Definition 2.1 *Correlation functions are defined by*

$$W_{k+1}^{(h)}(p, \mathbf{p_K}) := \sum_i \operatorname{Res}_{q \rightarrow \alpha_i} \frac{dE_q^{(i)}(p)}{(y(q) - y(\bar{q}))dx(q)} \left[\sum_{m=0}^h \sum_{J \subset K}^{\prime} W_{j+1}^{(m)}(q, \mathbf{p_J}) W_{k-j+1}^{(h-m)}(\bar{q}, \mathbf{p_{K \setminus J}}) + W_{k+2}^{(h-1)}(q, \bar{q}, \mathbf{p_K}) \right] \quad (2-15)$$

where \sum^{\prime} in the RHS means that we exclude the terms with $(m, J) = (0, \emptyset)$, and (h, K) .

We also define the *symplectic invariants*

Definition 2.2 *For $h \geq 2$, the genus h symplectic invariant is given by*

$$\mathcal{F}^{(h)} := \frac{1}{2-2h} \sum_i \operatorname{Res}_{q \rightarrow \alpha_i} \Phi(q) W_1^{(h)}(q) \quad (2-16)$$

where $\Phi(q)$ is any primitive of ydx , whereas for $h = 0, 1$, they are given by

$$\mathcal{F}^{(1)} := -\frac{1}{2} \ln(\tau_{Bx}) - \frac{1}{24} \ln \left(\prod_i y'(a_i) \right) \quad (2-17)$$

where τ_{Bx} is the Bergmann τ -function defined in [11, 22] and

$$\mathcal{F}^{(0)} := \frac{1}{2} \sum_i \operatorname{Res}_{a_i} V_i y dx + \frac{1}{2} t_{0,i} \mu_i - \frac{1}{4i\pi} \sum_i \oint_{\mathcal{A}_i} y dx \oint_{\mathcal{B}_i} y dx \quad (2-18)$$

where

$$V_i(p) := \operatorname{Res}_{q \rightarrow \alpha_i} y(q) dx(q) \ln \left(1 - \frac{z_i(p)}{z_i(q)} \right) \quad (2-19)$$

and

$$\mu_i := \int_{\alpha_i}^o \left(y dx dV_i + t_{0,i} \frac{dz_i}{z_i} \right) + V_i(o) - t_{0,i} \ln(z_i(o)). \quad (2-20)$$

The correlation functions and free energies can be seen as the terms of the topological expansions of some complete functions depending on an extra-variable N :

Definition 2.3 *Let the complete correlation functions and free energies be defined by*

$$\mathcal{W}_k(p_1, \dots, p_k) := \sum_{h=0}^{\infty} N^{2-2h-k} W_k^{(h)}(p_1, \dots, p_k) \quad (2-21)$$

and

$$\mathcal{F}(\mathcal{E}) := \sum_{h=0}^{\infty} F^{(2-2h)}(\mathcal{E}). \quad (2-22)$$

Let the partition function associated to the spectral curve $\mathcal{E}(x, y)$ be

$$\mathcal{Z}(\mathcal{E}) := e^{-\mathcal{F}(\mathcal{E})}. \quad (2-23)$$

2.2.2 Variation wrt the moduli of the spectral curve

These free energies and thus the partition functions are functions of the moduli of the algebraic curve \mathcal{E} . Among all the properties of these functions, it is interesting to note that their variations with respect to the moduli follow a simple rule (see [14]):

Theorem 2.1 *When the one form $y dx$ changes infinitesimally to $y dx(p) + \epsilon \Omega(p)$ with*

$$\Omega(p) = \int_{\partial\Omega} \Lambda(q) B(p, q) \quad (2-24)$$

for some function Λ and an integration contour $\partial\Omega$ away from the branch points, the correlation functions and free energies change as follows

$$W_k^{(h)}(\mathbf{pK}) \rightarrow W_k^{(h)}(\mathbf{pK}) + \epsilon \int_{\partial\Omega} \Lambda(q) W_{k+1}^{(h)}(q, \mathbf{pK}) \quad (2-25)$$

and

$$F^{(h)} \rightarrow F^{(h)} + \epsilon \int_{\partial\Omega} \Lambda(q) W_1^{(h)}(q). \quad (2-26)$$

Since the moduli of the curve are encoded in the one form $y dx$, one can extract from this theorem the variation of the correlation functions and free energies wrt them (see [14] for details).

More precisely, the correlation functions themselves were built as the result of a particular variation of the spectral curve changing $y dx(p)$ to $y dx(p) + \epsilon B(p, q)$. This variation is encoded in the so-called *loop insertion operator* $\partial_{B(\cdot, p)}$ defined by

$$\partial_{B(\cdot, p)} y dx(q) = B(p, q). \quad (2-27)$$

This operator, depending on one point p of the spectral curve, can be used to summarize the variations of all the moduli of the spectral curve at once by looking at its Taylor expansion around a singularity of the spectral curve. This is the basis of the arising of the Virasoro constraints studied in the forthcoming sections.

3 Global and local Virasoro constraints

In this section, we show that the partition function $\mathcal{Z}(\mathcal{E})$ associated to a given algebraic curve \mathcal{E} is the zero mode of two operator valued meromorphic 1-forms on the considered algebraic curve: the *global Virasoro operators*. More precisely, one has one differential operator associated to the moduli of the spectral curve at the poles of ydx whereas the other one involves the moduli at the x -branch points. Moreover the first operator is equivalent to the so-called loop equations whereas the second one is equivalent to the recursive solution of [14] defining the correlation functions.

3.1 Loop equations and global Virasoro constraints from the poles

In [14], the correlation functions associated to a given algebraic equation were related to the variations of the free energy with respect to the moduli of this equation chosen to be the coefficients of the Taylor expansion of the 1-form ydx near its poles. In the present section, we go further, showing that one can extract from this information a general differential equation mimicking the loop equations derived in the context of the random matrix theory. Indeed, the correlation functions can be shown to satisfy some similar "loop equations":

Theorem 3.1 *For any set of points $\{p, p_1, \dots, p_k\} \in \Sigma^{k+1}$, the complete correlation functions $W_k(p_1, p_2, \dots, p_k) := \sum_g N^{2-2g-k} W_k^{(g)}(p_1, \dots, p_k)$ satisfy the loop equations:*

$$\sum_{l=0}^k W_{l+1}(p, \mathbf{p}_L) W_{k-l+1}(p, \mathbf{p}_{K \setminus L}) + \frac{1}{N^2} W_{k+2}(p, p, \mathbf{p}_K) = P_{1,k}(p, \mathbf{p}_K) dx(p)^2 \quad (3-1)$$

where the function

$$P_{1,k}(p, \mathbf{p}_K) := \sum_i \oint_{\alpha_i} \frac{\sum_{l=0}^k W_{l+1}(q, \mathbf{p}_L) W_{k-l+1}(q, \mathbf{p}_{K \setminus L}) + \frac{1}{N^2} W_{k+2}(q, q, \mathbf{p}_K)}{(z_i(p) - z_i(q)) dx(q)} \quad (3-2)$$

is a function of p with poles only at the poles of ydx .

proof:

It follows from the loop equations derived in the simplest mixed case in [15]. Consider the loop equations (3-29) of [15] divided by $H_{0,0}(p, q)$ and take the residue as q approaches the poles β_i of xdy . The function $P_{1,k}(p, \mathbf{p}_K)$ is then equal to $\text{Res}_{q \rightarrow \beta_i} \frac{\tilde{U}_{k,0}(x(p), q; \mathbf{p}_K) dy(q)}{H_{0,0}(p, q)}$. The formula comes directly from the pole decomposition of the function $P_{1,k}(p, \mathbf{p}_K)$.

□

As it was pointed out in [14], the correlation functions can be seen as variations of the free energy when one changes the moduli of the spectral curve. Remember that one can see the correlation functions as the result of the action of the loop insertion operator on the free energy:

$$\partial_{B(.,q)} \mathcal{F} = W_1(q) \quad (3-3)$$

and

$$\partial_{B(.,p_1)} \partial_{B(.,p_2)} \mathcal{F} = W_2(p_1, p_2) \quad (3-4)$$

where the free energy is the sum of all the genus contributions: $\mathcal{F} := \sum_g N^{2-2g} F^{(g)}$. It is also important to remark that this loop insertion operator is just build to deal with the variations of all the moduli at the poles at once. Thanks to these properties one can show that the partition function $\mathcal{Z} = e^{-\mathcal{F}}$ is the solution of a differential equation involving the moduli of the spectral curve.

Theorem 3.2 *For any point $p \in \Sigma$, the partition function satisfies*

$$\mathcal{L}(p)\mathcal{Z} = 0 \quad (3-5)$$

where one defines the global Virasoro operator⁷

$$\mathcal{L}(p) := \frac{1}{N^2} : \mathcal{J}^2(p) : + \sum_i \oint_{\alpha_i} \frac{: \mathcal{J}^2(q) :}{(z_i(q) - z_i(p))dx(q)} \quad (3-6)$$

where the current is defined on any point of the spectral curve by

$$\mathcal{J}(p) := N y dx(p) + \frac{1}{N} \partial_{B(\cdot, p)}. \quad (3-7)$$

proof:

It directly follows from the properties Eq. (3-3) and Eq. (3-4) and the loop equation. \square

The partition function is thus the zero mode of a global operator defined as an operator valued meromorphic differential on the spectral curve. This operator, labeled by a point on the spectral curve, is in fact used to summarize different variations of the partition functions wrt to the moduli of the spectral curve on its poles, i.e. the moduli of the underlying theory.

3.2 Recursive relations and global constraints from the branch points

In the preceding section, we used a detour to build a global operator on the spectral curve annihilating the partition function in order to work with the moduli introduced in [14]. Indeed, we first built some loop equations satisfied by the correlation functions in order to build this global operator thanks to its projection around the poles of ydx . But we could choose to describe the 1-form ydx by its behavior at its zeroes instead of its poles, introducing moduli of the spectral curve as the coefficients of the taylor expansion of ydx around the x -branch points.

Let us consider the definition of $W_1^{(g)}$ for any $g > 0$:

$$W_1^{(g)}(p) = \sum_i \text{Res}_{q \rightarrow a_i} \frac{dE_q^{(i)}(p)}{(y(q) - y(\bar{q}))dx(q)} \left[\sum_{h=1}^{g-1} W_1^{(h)}(q) W_1^{(g-h)}(\bar{q}) + W_2^{g-1}(q, \bar{q}) \right]. \quad (3-8)$$

One can remark that the left hand side can be included in the RHS by writing⁸:

$$W_1^{(g)}(p) = \sum_i \text{Res}_{q \rightarrow a_i} \frac{dE_q^{(i)}(p)}{(y(q) - y(\bar{q}))dx(q)} \left(ydx(q) W_1^{(g)}(\bar{q}) + ydx(\bar{q}) W_1^{(g)}(q) \right). \quad (3-9)$$

After summing over the genus g and writing the correlation functions in terms of the variations of the partition function one get

⁷The name Virasoro operator is related to the projection of this operator around the poles (cf section 4).

⁸For detailed computations, refer to equation (4-12) of [9].

Theorem 3.3 *For any point p on the spectral curve, the partition function is a zero mode of the global Virasoro operator $\widehat{\mathcal{L}}(p)$*

$$\widehat{\mathcal{L}}(p)\mathcal{Z} = 0 \quad (3-10)$$

with

$$\widehat{\mathcal{L}}(p) := \sum_i \oint_{a_i} \frac{dE_q^{(i)}(p)}{(y(q) - y(\bar{q}))dx(q)} : \mathcal{J}(q)\mathcal{J}(\bar{q}) : . \quad (3-11)$$

We have thus built a second global Virasoro operator on the spectral curve. Note that the first one Eq. (3-6) was defined in terms of an integral around the poles of ydx while the new one is given in terms of a contour around the zeroes of this differential form. Its is also interesting to note that they carry the same form thanks to the following lemma

Lemma 3.1 *The global Virasoro operator Eq. (3-11) can be written*

$$\begin{aligned} \widehat{\mathcal{L}}(p) &:= - \sum_i \oint_{a_i} \frac{dE_q^{(i)}(p)}{(y(q) - y(\bar{q}))dx(q)} : \mathcal{J}^2(q) : \\ &= \sum_i \widehat{\mathcal{L}}_i(p) \end{aligned} \quad (3-12)$$

with

$$\widehat{\mathcal{L}}_i(p) := - \oint_{a_i} \frac{dE_q^{(i)}(p)}{(y(q) - y(\bar{q}))dx(q)} : \mathcal{J}^2(q) : . \quad (3-13)$$

proof:

The proof relies on the properties of the correlation functions as one changes sheets. It can be built from the equations (4-5), (4-7), (4-14) and (4-15) in [9]. \square

4 Local Virasoro constraints

We have now defined two global operators annihilating the partition function and involving all the moduli of the spectral curve. Let us now project these constraints in the neighborhood of different singularities to make the link with the Virasoro algebra clear, i.e. we look at this differential equation in different regime in the moduli space of the spectral curve⁹.

4.1 Virasoro constraint at the poles and Hermitian one matrix model representation

When the argument of the loop insertion operator approaches a pole of ydx , one can expand the latter in terms of the local variable z_i :

$$\mathcal{J}(p) \sim \sum_{j=1}^{d_i} \frac{dz_i(p)}{z_i^{j+1}(p)} \frac{\partial}{\partial t_{j,i}} + \sum_j j t_{i,j} z_i^{j-1}(p) dz_i(p). \quad (4-1)$$

We can then project the global constraint Eq. (3-6) in the neighborhood of this pole and get:

Theorem 4.1 *For any point p in the neighborhood of a pole α_i of ydx :*

$$L_-^{(i)}(p)\mathcal{Z} = 0 \quad (4-2)$$

⁹It is interesting to see that moving the point in the spectral curve really corresponds to selecting a regime in the moduli space of this curve.

where the local Virasoro operator is defined as the loop operator

$$L_-^{(i)} := \oint_{\alpha_i} \frac{1}{(z_i(q) - z_i(p))dz_i(q)} : \tilde{J}^{(i)}(q)^2 : \quad (4-3)$$

with the current

$$J^{(i)}(p) := \sum_{k \geq 0} \left[\frac{kt_{k,i}}{2} z_i(p)^{k-1} dz_i(p) + \frac{dz_i(p)}{z_i(p)^{k+1}} \frac{\partial}{\partial t_{k,i}} \right]. \quad (4-4)$$

It can be convenient to write this operator as

$$L_-^{(i)}(p) = \sum_{j=0}^{+\infty} \frac{dz_i(p)}{z_i(p)^{j+1}} L_j^{(i)} \quad (4-5)$$

with the discrete Virasoro operators

$$L_j^{(i)} = \frac{1}{N^2} \left(2j \frac{\partial}{\partial t_{j,i}} + \sum_{l=1}^{j-1} l(j-l) \frac{\partial^2}{\partial t_{j-l,i} \partial t_{l,i}} \right) + \sum_{k=1}^{d_i} (k+j) t_{j,i} \frac{\partial}{\partial t_{k+j,i}}. \quad (4-6)$$

It is easily checked that they indeed satisfy the commutation relations

$$[L_j^{(i)}, L_l^{(k)}] = (j-l) L_{j+l} \delta_{i,k}. \quad (4-7)$$

Thus, the symplectic invariants are D -modules in the sense of [2, 3, 4] as they are solution to some Virasoro constraints. It is remarkable that one can associate one set of Virasoro constraints to each pole of ydx . One can actually consider these different Virasoro algebra as local realizations of the global constraints imposed by the global Virasoro operator 3-6.

It is also interesting to note that the Virasoro constraints associated to a pole of the algebraic curve appears explicitly in the study of the Hermitian one matrix model defined by the partition function:

$$\mathcal{Z}_H(\{t_k\}) := \int_{\mathcal{H}_N} dM e^{N \sum_{k=0}^d t_k \text{Tr } M^k} \quad (4-8)$$

where one integrates over $N \times N$ Hermitian matrices M .

Indeed, in this case, the partition function is the symplectic invariant built from the spectral curve:

$$\mathcal{E}_H(x, y) := y^2 - \left(\sum_k kt_k x^{k-1} \right)^2 + P(x) \quad (4-9)$$

where $P(x)$ is a polynomial of degree at most $d-1$. Thus, the function y has two poles: one simple pole noted ∞_y and one pole of degree d denoted by ∞_x . Moreover, x has a simple pole at ∞_x and one can precise the behavior of ydx :

$$ydx(p) \sim \sum_k kt_k x^{k-1}(p) dx(p) \quad \text{as} \quad p \rightarrow \infty_x. \quad (4-10)$$

We are thus in the case described in this section with the t_i of the decomposition in the neighborhood of the pole given by the coefficients of the polynomial action in the matrix integral. Thus, one gets:

Theorem 4.2 *The partition function \mathcal{Z}_H is solution of the Virasoro constraints:*

$$L_j^{(i)} \mathcal{Z}_H(\{t_k\}) = 0 \quad , \quad \text{for } j \geq 0. \quad (4-11)$$

Since the t_k 's involved in this equation are the only parameters of this model, these equation, and thus the neighborhood of this unique pole of ydx , are sufficient to describe this model. This is why this representation of the free energy is often used when one encounters this type of Virasoro constraints: typically, around a pole of ydx (or xdy) a D-module can be represented under this form.

It means that in this regime where one plugs in only the moduli at one pole, the partition function reduces to the one of the hermitian 1 matrix model.

4.2 Virasoro constraints at the branch points and Kontsevich integral

One can also build such operators in the vicinity of the branch points a_i giving rise to the Kontsevich kernel by the gaussian case decomposition.

For this purpose, one has to expand the one-form ydx in the neighborhood of its zeroes, i.e. the branch points, to emphasize the moduli at the branch points. We thus have to introduce local parameterizations in the vicinity of the branch points a_i . Since one considers only simple branch points, one has a natural parameter in the vicinity of a branch point a_i :

$$\hat{z}_i(p) := \sqrt{x(p) - x(a_i)}. \quad (4-12)$$

When $p \rightarrow a_i$:

$$\hat{z}_i(p) \sim y(p) - y(a_i) \quad (4-13)$$

thus

$$y(p)dx(p) \sim 2y(a_i)\hat{z}_i(p)d\hat{z}_i(p) + 2\hat{z}_i^2(p)d\hat{z}_i(p). \quad (4-14)$$

More precisely, let us write down the Taylor expansion of the 1-form ydx in the neighborhood of the branch point a_i in terms of the local variable $z_i(p)$:

$$ydx(p) = \sum_{j=2}^{\infty} \tau_{j,i} \hat{z}_i(p)^{j-1} d\hat{z}_i(p) \quad (4-15)$$

with

$$\tau_{j,i} := \text{Res}_{p \rightarrow a_i} y(p)dx(p) \hat{z}_i^{-j-1}(p). \quad (4-16)$$

One can now blow up the spectral curve around this branch point, by expressing it in terms of the local coordinate \hat{z}_i . The blown up spectral curve admits a rational parameterization:

$$\begin{cases} \tilde{x}(z) = z^2 \\ \tilde{y}(z) = \sum_{k=2}^{\infty} \tau_{k,i} z^{k-2} \end{cases} \quad (4-17)$$

It was proved in [14] that the local behavior of the symplectic invariants around a critical point is given by the symplectic invariants of the blown up spectral curve. Thus, in the vicinity of the branch points a_i , the symplectic invariants reduce to those of Eq. (4-17) which are given by the topological expansion of the Kontsevich integral defined as follows: [23, 21]

$$\mathcal{Z}_K(\tau_{k,i}) := \int dM \, e^{-N \text{Tr} (\frac{M^3}{3} - M(\Lambda^2 + \tau_1))} \quad , \quad \tau_1 = \frac{1}{N} \text{Tr} \frac{1}{\Lambda} \quad (4-18)$$

where Λ is a deterministic external matrix defined by

$$\tau_{k,i} = \frac{1}{N} \text{Tr} \Lambda^{-k}. \quad (4-19)$$

This matrix integral is known to satisfy continuous Virasoro constraints in terms of the $\tau_{j,i}$:

$$\forall j \geq 2, \widehat{L}_j^{(i)} \mathcal{Z}_K(\tau_{k,i}) = 0 \quad (4-20)$$

where the operator $\widehat{L}_j^{(i)}$ can be found in [2]

Hence, the partition function \mathcal{Z} satisfies:

$$\forall j \geq 2, \forall i, \widehat{L}_j^{(i)} \mathcal{Z} = 0. \quad (4-21)$$

5 Givental like decomposition

From the first section, one knows that there exists two families of Virasoro operators defined in the neighborhood of the poles and the zeros respectively annihilating the global partition function:

$$\left\{ \begin{array}{l} \left[\sum_i \mathcal{L}_i(p) + \mathcal{L}(p) \right] \mathcal{Z} = 0 \\ \sum_i \widehat{\mathcal{L}}_i(p) \mathcal{Z} \end{array} \right. . \quad (5-1)$$

It means that one can decompose the partition function \mathcal{Z} in two ways:

- it is the product of zeros-modes of the operators $\mathcal{L}_i(p)$, which are nothing but the partition functions of the one matrix model studied in section 4.1;
- it is the product of the zero-modes of the operators $\widehat{\mathcal{L}}_i(p)$ which are nothing but Kontsevich integrals studied in section 4.2.

This means that the partition function \mathcal{Z} can be decomposed as a product of 1 matrix model integrals or Kontsevich integrals (which are KdV tau-functions) up to some conjugation operator mixing the local variables at the branch points and poles of ydx . This reproduces the decomposition formulae discovered by Givental for multi-component KP tau functions [19, 20]. Let us write these two types of decomposition explicitly.

Nevertheless, it must be noted that the Virasoro constraints and differential equations studied so far only involve one part of the moduli of the spectral curve: they do not care about the filling fractions ϵ_i . Thus, if the spectral curve has non-vanishing genus, one should fix the dependence of both sides of the decomposition formula on these filling fractions in order to get the right equality. This point is still under investigations and we consider in the following of this paper that the spectral curve has genus 0.

5.1 Decomposition of the global partition function in local partition functions

Let us consider a prototype of Givental's like decomposition, i.e. a decomposition of the global Virasoro operator as a product of local Virasoro operators in the neighborhood of a set of singular point ξ_i :

$$\mathcal{L}(p) := \sum_i \oint_{\xi_i} \frac{dz_i(q)}{z_i(p) - z_i(q)} : \mathcal{J}(q)^2 : \quad (5-2)$$

with

$$\mathcal{J}(q) = Nydx(q) - \frac{1}{N} \partial_{B(.,q)} \quad (5-3)$$

and the inverse of a local variable z_i in the neighborhood of ξ_i .

Let us suppose that the global current $\mathcal{J}(p)$ converges to local currents $\mathcal{J}_i(p)$ as $p \rightarrow \xi_i$ where

$$\mathcal{J}_i(p) = N \sum_k k t_{k,i} z_i^{k-1}(p) dz_i(p) + \frac{1}{N} \frac{dz_i(p)}{z_i^{k+1}(p)} \frac{\partial}{\partial t_{k,i}} \quad (5-4)$$

for some fixed times $t_{k,i}$. We now decompose the zero mode \mathcal{Z} of the global Virasoro operator

$$\mathcal{L}(p)\mathcal{Z} = 0 \quad (5-5)$$

into a product of the zero modes of the local operators:

$$\mathcal{L}_i(p)\mathcal{Z}_i = 0 \quad (5-6)$$

thanks to some conjugation operator \mathcal{U} :

$$\mathcal{Z} = e^{\mathcal{U}} \prod_i \mathcal{Z}_i. \quad (5-7)$$

In order to compute the conjugation operator \mathcal{U} , one first identify the difference between the local currents and the global one:

$$\Delta\mathcal{J}_i(p) = \mathcal{J}(p) - \mathcal{J}_i(p), \quad (5-8)$$

since the intertwining operator is build to compensate this difference.

For this purpose, following [3], let us define the bi-differential $f_{\mathcal{O},\mathcal{O}'}(p,p')$ associated to operators $\mathcal{O}(p)$ and $\mathcal{O}'(p')$:

$$f_{\mathcal{O},\mathcal{O}'}(p,p') := \mathcal{O}(p)\mathcal{O}'(p') - : \mathcal{O}(p)\mathcal{O}'(p') :. \quad (5-9)$$

Especially, one has

$$f_{\mathcal{J},\mathcal{J}}(p,p') = \frac{1}{N^2} B(p,p') \quad (5-10)$$

and

$$f_{\mathcal{J}_i,\mathcal{J}_i}(p,p') = \frac{1}{N^2} \frac{dz_i(p)dz_i(p')}{(z_i(p) - z_i(p'))^2}. \quad (5-11)$$

One can compute explicitly

$$f_{\mathcal{J},\mathcal{J}}(p,p') - f_{\mathcal{J}_i,\mathcal{J}_i}(p,p') = \sum_{k,l} A_{k,l}^{(i)} z_i^k(p) z_i^l(p'). \quad (5-12)$$

On the other hand, one has

$$\begin{aligned} f_{\mathcal{J},\mathcal{J}}(p,p') - f_{\mathcal{J}_i,\mathcal{J}_i}(p,p') &= \mathcal{J}(p)\mathcal{J}(p') - : \mathcal{J}(p)\mathcal{J}(p') : - \mathcal{J}_i(p)\mathcal{J}_i(p') + : \mathcal{J}(p)\mathcal{J}_i(p') : \\ &= [\mathcal{J}_i(p) + \Delta\mathcal{J}_i(p)] [\mathcal{J}_i(p') + \Delta\mathcal{J}_i(p')] - : [\mathcal{J}_i(p) + \Delta\mathcal{J}_i(p)] [\mathcal{J}_i(p') + \Delta\mathcal{J}_i(p')] : \\ &\quad - \mathcal{J}_i(p)\mathcal{J}_i(p') + : \mathcal{J}(p)\mathcal{J}_i(p') : \\ &= \Delta\mathcal{J}_i(p)\mathcal{J}_i(p') - : \Delta\mathcal{J}_i(p)\mathcal{J}_i(p') :. \end{aligned} \quad (5-13)$$

Looking for a generic solution of the form

$$\Delta\mathcal{J}_i(p) = \sum_{k,l} c_{k,l}^{(i)} z_i^k(p) \frac{\partial}{\partial t_{l,i}}, \quad (5-14)$$

one gets

$$\Delta\mathcal{J}_i(p) = \sum_{k,l} A_{k,l}^{(i)} z_i^k(p) \frac{\partial}{\partial t_{l,i}}. \quad (5-15)$$

This can be written

$$\Delta \mathcal{J}_i(p) = \sum_j \oint_{\xi_j} A^{(i,j)}(p, q) \Omega_j(q) \quad (5-16)$$

where

$$A^{(i,j)}(p, p') := f_{\mathcal{J}, \mathcal{J}}(p, p') - f_{\mathcal{J}_i, \mathcal{J}_j}(p, p') \quad (5-17)$$

and

$$\Omega_j(p) := N \sum_k t_{k,i} z_i^k(p) dz_i(p) - \frac{1}{N} \frac{dz_i(p)}{k z_i^k(p)} \frac{\partial}{\partial t_{k,i}}. \quad (5-18)$$

Finally, the conjugation operator is constrained by

$$\forall i, \Delta \mathcal{J}_i(p) = [\mathcal{J}_i(p), \mathcal{U}]. \quad (5-19)$$

This set of equations admits as solution

$$\boxed{\mathcal{U} = \sum_{i,j} \oint_{\xi_j} \oint_{\xi_i} A^{(i,j)}(p, q) \Omega_j(q) \Omega_i(p).} \quad (5-20)$$

Let us apply this analysis to the decomposition in one hermitian matrix model partition functions and Kontsevich integrals.

5.2 Virasoro at poles: decomposition in 1 matrix models

Let us first consider the case of the poles of ydx : $\{\xi_i\} := \{\alpha_i\}$. From Eq. (4-1), one knows that the global current $\mathcal{J}(p)$ tends to the local current $\mathcal{J}_i(p)$ as $p \rightarrow \alpha_i$. On the other hand, in section 4.1, we proved that the partition function of the one hermitian matrix model

$$\mathcal{Z}_H(\mathbf{t}) = \int_{\mathcal{H}_N} dM e^{-N \sum_{k=0}^d t_k \text{Tr } M^k} \quad (5-21)$$

is a zero mode of the local Virasoro operator

$$\mathcal{L}_i(p)(\mathbf{t}) \mathcal{Z}_H(\mathbf{t}) = 0. \quad (5-22)$$

The previous section implies then that

Theorem 5.1 *The global partition function can be decomposed in a product of one hermitian matrix integrals associated to the poles α_i of the meromorphic form ydx*

$$\boxed{\mathcal{Z}(\mathbf{t}_1, \mathbf{t}_2, \dots) = e^{\mathcal{U}} \prod_i \mathcal{Z}_H(\mathbf{t}_i).} \quad (5-23)$$

with the intertwining operator \mathcal{U} defined by

$$\mathcal{U} := \sum_{i,j} \oint_{\alpha_j} \oint_{\alpha_i} A^{(i,j)}(p, q) \Omega_j(q) \Omega_i(p) \quad (5-24)$$

where

$$A^{(i,j)}(p, q) := f_{\mathcal{J}, \mathcal{J}}(p, p') - f_{\mathcal{J}_i, \mathcal{J}_j}(p, p')$$

$$= B(p, q) - \frac{dz_i(p)dz_j(q)}{(z_i(p) - z_j(q))^2} \quad (5-25)$$

and

$$\Omega_i(p) := N \sum_k t_{k,i} z_i^k(p) dz_i(p) - \frac{1}{N} \frac{dz_i(p)}{k z_i^k(p)} \frac{\partial}{\partial t_{k,i}}. \quad (5-26)$$

5.3 Virasoro at branch points: decomposition in Kontsevich integrals

Let us now consider the case of the branch points: $\{\xi_i\} := \{a_i\}$.

In section 4.2, we proved that the Kontsevich integral

$$\mathcal{Z}_K := \int dM e^{-N \text{Tr} (\frac{M^3}{3} - M(\Lambda^2 + \tau_1))} \quad , \quad \tau_1 = \frac{1}{N} \text{Tr} \frac{1}{\Lambda} \quad (5-27)$$

is a zero mode of the local Virasoro operator

$$\hat{\mathcal{L}}_i(p)(\tau) \mathcal{Z}_K(\tau) = 0. \quad (5-28)$$

One thus has the decomposition formula

Theorem 5.2 *The global partition function can be decomposed in a product of Kontsevich integrals associated to the branch points a_i :*

$$\boxed{\mathcal{Z}(\tau_1, \tau_2, \dots) = e^{\hat{\mathcal{U}}} \prod_i \mathcal{Z}_K(\tau_i)} \quad (5-29)$$

with the intertwining operator $\hat{\mathcal{U}}$ defined by

$$\hat{\mathcal{U}} := \sum_{i,j} \oint_{a_j} \oint_{a_i} \hat{A}^{(i,j)}(p, q) \hat{\Omega}_j(q) \hat{\Omega}_i(p) \quad (5-30)$$

where

$$\begin{aligned} \hat{A}^{(i,j)}(p, q) &:= f_{\mathcal{J}, \mathcal{J}}(p, p') - f_{\hat{\mathcal{J}}_i, \hat{\mathcal{J}}_j}(p, p') \\ &= B(p, q) - \frac{d\hat{z}_i(p) d\hat{z}_j(q)}{(\hat{z}_i(p) - \hat{z}_j(q))^2} \end{aligned} \quad (5-31)$$

and

$$\hat{\Omega}_i(p) := N \sum_k \tau_{k,i} \hat{z}_i^k(p) d\hat{z}_i(p) - \frac{1}{N} \frac{d\hat{z}_i(p)}{k \hat{z}_i^k(p)} \frac{\partial}{\partial \tau_{k,i}}. \quad (5-32)$$

6 Conclusion and perspectives

In this paper, we investigated and made precise the link between the definition of the partition function of [3] as a D-module and the recursive one of [14]. Indeed, we proved that the recursive definition of the correlation functions of [14] are nothing but Virasoro constraints located at the branch points of the spectral curve whereas the loop equations initially solved in the matrix models context can be seen as Virasoro constraints localized at the poles of the one form ydx on the spectral curve. This means that both approaches are totally equivalent as it was already pointed out in [26] and thus the

symplectic invariants are the string theory partition function studied in [3]. We also completed the work of [3] by studying not only hyperelliptical curves but generic spectral curves. Moreover, we have pointed out that the duality between the one hermitian matrix model and the Kontsevich integral just follows from Virasoro constraints on the poles and on the branch points of the same spectral curve. Finally, one was led to decompose the partition function as a product of random matrix integrals: 1 hermitian random matrix integrals at the poles and Kontsevich integrals at the branch points. This gives a nice new representation of the symplectic invariants and a formalism complementary to the approach of [14].

It is interesting to note that these decomposition formulae were already derived by Givental to express the multi-component KP tau-function as a product of KdV tau-functions (i.e. the Kontsevich integral). It was already mentioned in [14] that the partition function build from the symplectic invariants could also be defined as the tau-function of an integrable model (as it is proved for some matrix model's cases). The arising of the Givental decomposition formulae points also in this direction and the formalism borrowed from [3] can be very useful to derive properly a Hirota equation which can be understood as a defining property of the multi-component KP-tau function (e.g. see [1]), but this is left to a forth-coming work.

Moreover, as it was already remarked, we only focused in this paper on the moduli at the singularities of the spectral curve and left the filling fractions aside. Nevertheless, it should be possible to study this other type of moduli in the same way using the variation of the partition function wrt them. This step is fundamental if one wants to obtain Givental like decomposition formulae for higher genus spectral curve and deserves further investigations.

Among the numerous possible applications of the symplectic invariants, let us mention two of them which can benefit directly from this formalism and would merit further investigations. First, we only considered the "non-mixed correlation functions"¹⁰ and the related loop equations. From this restricted set of observables, one was able to extract Virasoro constraints which can be seen as a restriction of the more general \mathcal{W} -algebra constraints encountered for example in the 2-hermitian matrix models [25]. For this purpose, it should be possible to express the mixed correlation functions of [13], generalized away from the matrix models [6], as the result of the action of a differential operator on the partition function: these new operators should form some \mathcal{W} -algebra. Another aspect which should benefit investigations is the link between this formalism and Krichever-Novikov like algebras [24]. Indeed, as it is pointed out in [3], the currents \mathcal{J} should satisfy some commutation relations giving rise to a Krichever-Novikov algebra or, more precisely, it should give rise to its generalization to arbitrary Riemann surface by Schlichenmaier [28].

Acknowledgements:

I would like to thank Marc Adler, Mattia Cafasso, Bertrand Eynard and Pierre Van Moerbeke for fruitful discussions on the subject and particularly Leonid Chekhov for his comments on a first version of this paper. This work was supported by the ENRAGE European network MRTN-CT-2004-005616, by the ENIGMA European network MRT-CT-2004-5652, by the French and Japanese governments through PAI SAKURA, by the European Science Foundation through the MISGAM program and by

¹⁰This name is inherited from the study of multi-matrix model where these correlation functions are observables which do not mix the different random matrices.

the ANR project Géométrie Intégrabilité en Physique Mathématique ANR-BLAN-0029-01.

References

- [1] M.Adler, P.Van Moerbeke, P.Vanhaecke, “Moment matrices and multi-component KP, with application to random matrix theory”, math-ph/0612064.
- [2] A.Alexandrov, A.Mironov and A.Morozov, “ M-Theory of Matrix Models”, *Theor. Math. Phys.* **150** (2007) 179, hep-th/0605171.
- [3] A.Alexandrov, A.Mironov and A.Morozov, “Instantons and Merons in Matrix Models”, *Physica* **D235** (2007) 126, hep-th/0608228.
- [4] A.Alexandrov, A.Mironov and A.Morozov, “Partition function of matrix models as the first special functions of string theory. I. Finite size Hermitean 1-matrix model, *Int. J. Mod. Phys. A* **19** (2004) 4127, *Theor. Math. Phys.* **142** (2005) 349, hep-th/0310113.
- [5] J.Amnrjorn, C.F.Kristjansen, “From 1-matrix model to Kontsevich model”, *Mod. Phys. Lett.* **A8** (1993) 2875, hep-th/9307063.
- [6] L.Cantini, B.Eynard, N.Orantin, in preparation.
- [7] L.Chekhov, “Matrix Models and Geometry of Moduli Spaces”, hep-th/9509001.
- [8] L.Chekhov, B.Eynard, “Hermitian matrix model free energy: Feynman graph technique for all genera”, *J. High Energy Phys.* **JHEP03** (2006) 014, hep-th/0504116.
- [9] L.Chekhov, B.Eynard and N.Orantin, “Free energy topological expansion for the 2-matrix model”, *J. High Energy Phys.* **JHEP12** (2006) 053, math-ph/0603003.
- [10] B.Eynard, “Topological expansion for the 1-hermitian matrix model correlation functions”, JHEP/024A/0904, hep-th/0407261.
- [11] B.Eynard, A.Kokotov, D.Korotkin, “ $1/N^2$ corrections to free energy in Hermitian two-matrix model”, hep-th/0401166.
- [12] B.Eynard, N.Orantin, “Topological expansion of the 2-matrix model correlation functions: diagrammatic rules for a residue formula”, *J. High Energy Phys.* **JHEP12**(2005)034, math-ph/0504058.
- [13] B. Eynard, N. Orantin, “Topological expansion and boundary conditions” *JHEP* **06** (2008) 037, arXiv:0710.0223.
- [14] B.Eynard, N.Orantin, “Invariants of algebraic curves and topological expansion”, *Communication in Number Theory and Physics* **vol.1 n2**, math-ph/0702045.
- [15] B.Eynard, N.Orantin, “Topological expansion of mixed correlations in the hermitian 2 Matrix Model and x-y symmetry of the Fg invariants”, *J. Phys. Math. Theor.* **A41** (2008) 015203, arXiv:0705.0958.
- [16] B.Eynard, A.Prats Ferrer, “Topological expansion of the chain of matrices”, arXiv:0805.1368.
- [17] H.M. Farkas, I. Kra, “Riemann surfaces” 2nd edition, Springer Verlag, 1992.

- [18] J.D. Fay, “Theta functions on Riemann surfaces”, Springer Verlag, 1973.
- [19] A. Givental, “Semisimple Frobenius structure at higher genus”, *Intern. Math. Res. Notices* **23** (2001) 1265, math/0008067.
- [20] A. Givental, “ A_{n-1} -singularities and nKdV hierarchies”, *Moscow. Math Jour.* **3(2)** (2003) 475 , math.AG/0209205.
- [21] C.Itzykson, J.B.Zuber, “Combinatorics of the Modular Group II : the Kontsevich integrals”, *Int. J. Mod. Phys. A* **7** (1992) 5661, hep-th/9201001.
- [22] A.Kokotov, D.Korotkin, “Bergmann tau-function on Hurwitz spaces and its applications”, math-ph/0310008.
- [23] M.Kontsevitch, “Intersection theory on the moduli space of curves and the matrix Airy function”, *Funk. Anal. Prilozh.* **25** (1991) 50-57; Max-Planck Institut preprint MPI/91-47, MPI/91-77.
- [24] I. Krichever, S.P. Novikov, “Algebras of Virasoro type, Riemann surfaces and structure of the theory of solitons”, *Funct. Anal. Appl.* **21** (1987) 126; *Funct. Anal. Appl.* **21** (1987) 294.
- [25] A.Marshakov, A.Mironov, A.Morozov, “From Virasoro constraints in Kontsevich’s model to W-constraints in the 2-matrix model”, *Mod. Phys. Lett. A* **7** (1992) 1345, hep-th/9201010.
- [26] A.Mironov, A.Morozov, “Virasoro constraints for Kontsevich-Hurwitz partition function”, arXiv:0807.2843.
- [27] A. Morozov, “String theory: What is it?”, *Sov. Phys. Usp.* **35** (1992) 671.
- [28] M.Schlichenmaier, “Differential operator algebras on compact Riemann surfaces”, hep-th/9311036.